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We introduce the Britto-Cachazo-Feng-Witten (BCFW) recursion relation [1, 2] for tree amplitudes. This is a milestone of scattering amplitude development.

I. THREE-POINT GLUON TREE AMPLITUDE

Three-point gluon amplitude is a building block for BCFW recursion relation.

We consider the three-point massless kinematics. It is clear that

$$p_i \cdot p_j = 0, \forall 1 \leq i, j \leq 3. \quad (1)$$

or,

$$\langle 12 \rangle [12] = 0, \quad \langle 23 \rangle [23] = 0, \quad \langle 13 \rangle [13] = 0. \quad (2)$$

For real kinematics, $\langle ij \rangle$ is the complex conjugate of $[ij]$ and all spinor products are zero. It is impossible to study the amplitude in this case. However, we consider complex kinematics and the amplitude can be discussed.

We look at these kinematic conditions in details:

- $\langle 12 \rangle = 0$. In this case, we set $\lambda_2 = c \lambda_1$ where c is a complex number. As matrices,

$$\lambda_1 \tilde{\lambda}_1 + c \lambda_1 \tilde{\lambda}_2 = -\lambda_3 \tilde{\lambda}_3. \quad (3)$$

It implies that $\langle 13 \rangle = 0$ and $\langle 23 \rangle = 0$.

- $[12] = 0$. In this case, we have $[23] = [13] = 0$.

Then it is straightforward to compute the partial amplitudes from Feynman rules. In the case when $[12] = [23] = [13] = 0$, for the MHV amplitude, we can use

$$\epsilon_1^- = \sqrt{2} \frac{1\tilde{k}}{[1k]}, \quad \epsilon_2^- = \sqrt{2} \frac{2\tilde{k}}{[2k]}, \quad \epsilon_3^- = \sqrt{2} \frac{1\tilde{3}}{\langle 13 \rangle}, \quad (4)$$

where p^k is an arbitrary null vector. Then by straightforward computation, we find that the partial amplitude is

$$A(1^-, 2^-, 3^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}. \quad (5)$$

Here we used the fact that $\tilde{1}\tilde{1} + \tilde{2}\tilde{2} + \tilde{3}\tilde{3} = 0$ to get $\langle 12 \rangle [1k] + \langle 32 \rangle [3k] = 0$.

Similarly, in the case when $\langle 12 \rangle = \langle 23 \rangle = \langle 13 \rangle = 0$, for the $\overline{\text{MHV}}$ amplitude,

$$A(1^+, 2^+, 3^-) = -i \frac{[12]^4}{[12][23][31]}. \quad (6)$$

II. BCFW RELATION

For the n -point massless kinematics, we consider the BCFW shift,

$$p_k = \lambda_k \tilde{\lambda}_k \rightarrow p_k(z) \equiv \lambda_k (\tilde{\lambda}_k - z \tilde{\lambda}_n) \quad (7)$$

$$p_n = \lambda_n \tilde{\lambda}_n \rightarrow p_n(z) \equiv (\lambda_n + z \lambda_k) \tilde{\lambda}_n \quad (8)$$

and keep all the other momentum invariant. It is clear that

$$p_1 + \dots + p_k(z) + \dots + p_n(z) = 0 \quad (9)$$

and $p_k(z)^2 = p_n(z)^2 = 0$. So with an arbitrary value of z , we still have the n -point massless kinematics.

It is natural to consider the n -gluon tree amplitude $A(z)$ for the BCFW-shifted amplitude. Note that, by shifting p_k and p_n , we also shift the corresponding polarization vectors.

Because of the Feynman rules, $A(z)$ must be a rational function of z defined on the Riemann surface. Let $\{z_1, \dots, z_m\}$ be the pole locus of $A(z)$. Note that $0 \notin \{z_1, \dots, z_m\}$. Consider the meromorphic differential form $\omega = \frac{A(z)}{z} dz$. On the Riemann Sphere, ω may be divergent at

$$\{0, z_1, \dots, z_m, \infty\} \quad (10)$$

Global residue theorem ensures that,

$$A + \left(\sum_{i=1}^m \text{Res}_{z \rightarrow z_i} \frac{A(z)}{z} dz \right) + \text{Res}_{z \rightarrow \infty} \frac{A(z)}{z} dz = 0, \quad (11)$$

where $A(0) = A$ is the original gluon tree amplitude.

The key ingredient of BCFW shift is that for Yang-Mills theory, Gravity theory, super-Yang-Mills theory and supergravity theory, $A(z)$ has surprisingly simply behavior when $z \rightarrow \infty$. If the k -th gluon and n -th gluon have the helicities $(+, +)$, $(-, -)$ and $(-, +)$,

$$A(z) \sim 1/z, \quad z \rightarrow \infty \quad (12)$$

In these cases, the residue of ω vanishes at infinity. That implies,

$$A = - \sum_{i=1}^m \text{Res}_{z \rightarrow z_i} \frac{A(z)}{z} dz. \quad (13)$$

We have a close look at the $z \rightarrow z_i$. If $A(z)$ is divergent at $z \rightarrow z_i$, then the divergence must come from a propagator which depends on z . Let the divergent propagator be $-i\eta^{\mu\nu}/p(z)^2$ and $p(z_i)^2 = 0$. Note that $p(z_i)$ flow separates the color-ordered diagrams to two sub-diagrams. Call the sub-diagram with p_k as the left one, and the other sub-diagram with p_n as the right. We assume that $p(z_i)$ flow is from the left to right.

For the null vector $p(z_i)$, we can consider it as an on-shell gluon and define its polarization vectors. $\epsilon_+(p(z_i))$ and $\epsilon_-(p(z_i))$. Recall that

$$\epsilon_+^\mu(p(z_i))\epsilon_-^\nu(p(z_i)) + \epsilon_-^\mu(p(z_i))\epsilon_+^\nu(p(z_i)) = -\eta^{\mu\nu} + \frac{p(z_i)^\mu q^\nu + p(z_i)^\nu q^\mu}{p(z_i) \cdot q}. \quad (14)$$

The left part of the diagram is $M_{L,i,\mu}$ and the right part of the diagram is $M_{R,i,\mu}$. $M_{L,i,\mu}$, contracted with $\epsilon_+^\mu(p(z_i))$, becomes a tree amplitude $A_{L,i,+}$. $M_{R,i,\mu}$, contracted with $\epsilon_-^\mu(p(z_i)) = -\epsilon_-^\mu(-p(z_i))$, becomes a tree amplitude $A_{R,i,-}$. Because of the Ward identity, $p(z_i)\mu$ contractions vanish. Therefore,

$$\text{Res}|_{z \rightarrow z_i} \frac{A(z)}{z} = -i \sum_{h=\pm} \frac{A_{L,i,h} A_{R,i,-h}}{z_i} \lim_{z \rightarrow z_i} \frac{z - z_i}{p(z)^2} \quad (15)$$

Let the sum of ‘‘original’’ external momentum of the left diagram to be $P_{L,i}$. $p(z_i)^2 = P_{L,i}^2 - 2z_i P_{L,i} \cdot (k\tilde{n}) = 0$.

$$\text{Res}|_{z \rightarrow z_i} \frac{A(z)}{z} = i \sum_{h=\pm} \frac{A_{L,i,h} A_{R,i,-h}}{P_{L,i}^2} \quad (16)$$

and

$$A = -i \sum_i \sum_{h=\pm} \frac{A_{L,i,h} A_{R,i,-h}}{P_{L,i}^2}. \quad (17)$$

This is the BCFW recursion relation for gluon tree amplitudes. Note that $A_{L,i,h}$, $A_{R,i,-h}$ are both on-shell amplitudes. This is a great advantage since we do not need to worry about off-shell objects.

Gluon tree amplitude with the other helicity (+, -) does *not* vanish at infinity. For this configuration, we can cyclically permute the external legs and then use BCFW recursion relation.

III. APPLICATION

A. Four-point MHV

The goal is to calculate $A(1^-2^-3^+4^+)$. We consider the BCFW shift

$$p_1 = \lambda_1 \tilde{\lambda}_1 \rightarrow p_1(z) \equiv \lambda_1(\tilde{\lambda}_1 - z\tilde{\lambda}_4) \quad (18)$$

$$p_4 = \lambda_4 \tilde{\lambda}_4 \rightarrow p_4(z) \equiv (\lambda_4 - z\lambda_1)\tilde{\lambda}_4 \quad (19)$$

With a simple graphic analysis, we see that the only way the amplitude can diverge is that,

$$P^2(z) = (p_1 + p_2 - z1\tilde{4})^2 = s_{12} - \langle 12 \rangle [24] z \rightarrow 0 \quad (20)$$

In the BCFW language, we call this pole locus z_1 and $P_{L,1}^2 = s_{12}$. $z_1 = -[12]/[24]$. At this pole,

$$p_1(z_1) = 1(\tilde{1} + \frac{[12]}{[24]}\tilde{4}) = 1(\frac{[14]}{[24]}\tilde{2}) \quad (21)$$

$$p_4(z_1) = (4 - \frac{[12]}{[24]}1)\tilde{4} = (\frac{\langle 14 \rangle}{\langle 13 \rangle}3)\tilde{4} \quad (22)$$

$$P(z_1) = -p_1(z_1) - p_2 = -(\frac{[14]}{[24]}1 + 2)\tilde{2} = -\frac{\langle 12 \rangle}{\langle 13 \rangle}3\tilde{2} \quad (23)$$

where we used the Schouten identity and momentum conservation $\langle 13 \rangle [14] + \langle 23 \rangle [24] = 0$. Then we make a kinematics table for the left diagram,

$p_1(z_1)$	1	$\frac{[14]}{[24]}\tilde{2}$
p_2	2	$\tilde{2}$
$P(z_1)$	$-\frac{\langle 12 \rangle}{\langle 13 \rangle}3$	$\tilde{2}$

(24)

and for the right diagram,

p_3	3	$\tilde{3}$
$p_4(z_1)$	$\frac{\langle 14 \rangle}{\langle 13 \rangle}3$	$\tilde{4}$
$-P(z_1)$	$\frac{\langle 12 \rangle}{\langle 13 \rangle}3$	$\tilde{2}$

(25)

Note that for $p_1(z_1)$ and $p_4(z_1)$, we need to keep the spinor normalization: when $z \rightarrow 0$, $\lambda_1(z) \rightarrow \lambda_1$. We also have to carefully take the spinor normalization for $P(z_1)$ and $-P(z_1)$.

Left diagram corresponds to a $(--)$ MHV amplitude and the right corresponds to a $(++)$ $\overline{\text{MHV}}$ amplitude. By the BCFW recursion,

$$\begin{aligned} A(1^-2^-3^+4^+) &= \frac{-i}{s_{12}} \frac{\langle 12 \rangle^3}{\frac{\langle 12 \rangle^2}{\langle 13 \rangle^2} \langle 23 \rangle \langle 31 \rangle} \frac{[34]^3}{[42][23]} \\ &= i \frac{[34]^3}{[12][23][41]} = i \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \end{aligned} \quad (26)$$

B. n -point MHV amplitude

With BCFW recursion relation, it is straight forward to prove the n -point MHV amplitude with Parke-Taylor formula. However, historically, Parke-Taylor formula was proved by Berends-Giele recursion relations.

Without loss of generality, we consider the n -point tree amplitude $A(1^-, \dots, i^-, \dots, n^+)$, $i < n-1$. We choose the BCFW shift,

$$\begin{aligned} p_1(z) &= \lambda_1(\tilde{\lambda}_1 - z\tilde{\lambda}_n) \\ p_n(z) &= (\lambda_n + z\lambda_1)\tilde{\lambda}_n \end{aligned} \quad (27)$$

First we assume that p_1 is placed on the “left” and p_n is placed on the “right”. Consider two cases,

- p_i is placed on the “right”. By graph analysis, the only way to get nontrivial contribution is consider the left amplitude is 3-point and the right one is $(n-1)$ -point. Then the left amplitude should be $\overline{\text{MHV}}$ while the right amplitude should be MHV amplitude. However by a detailed analysis, we find the left graph has all right handed spinor in parallel. Therefore the left amplitude is zero, and does not contribute to the amplitude.
- p_i is placed on the “left”. By graph analysis, the only way to get nontrivial contribution is consider the left amplitude is $(n-2)$ -point and the right one is 3-point. Then the left amplitude should be MHV while the right amplitude should be $\overline{\text{MHV}}$ amplitude. The bridge,

$$P(z) = (n+z1)\tilde{n} \quad (28)$$

Assume that at $z \rightarrow z_1$ $P(z)^2 = 0$. Consider the right amplitude with $p_n, p_{n-1}, P(z)$. Its kinematics table is,

p_{n-1}	λ_{n-1}	$\tilde{\lambda}_{n-1}$	(29)
$p_n(z_1)$	$x\lambda_{n-1}$	$\tilde{\lambda}_n$	
$-P(z)$	λ_{n-1}	$-\tilde{\lambda}_{n-1} - x\tilde{\lambda}_n$	

It is easy to determine that $x = \langle 1n \rangle / \langle 1, n-1 \rangle$. This $\overline{\text{MHV}}$ amplitude is

$$A_R = -i \frac{[n-1, n]}{x}. \quad (30)$$

The right kinematic table is

$$\begin{array}{c|c|c}
P(z) & -\lambda_{n-1} & -\tilde{\lambda}_{n-1} - x\tilde{\lambda}_n \\
\hline
p_1(z) & 1 & \tilde{1} - z_1\tilde{\lambda}_n \\
\hline
\dots & \dots & \dots
\end{array} \tag{31}$$

This MHV amplitude is

$$A_L = i \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots (-1) \langle n-2, n-1 \rangle (-1) \langle n-1, 1 \rangle} \tag{32}$$

Combine them together, we proved the MHV amplitude formula for n -point,

$$A = \frac{-i}{s_{n,n-1}} A_L A_R = i \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}. \tag{33}$$

C. NMHV amplitude

The amplitudes $A(1^+, \dots i^-, \dots j^-, \dots k^-, \dots n^+)$ with $n \geq 6$ are called NMHV amplitude. The tree-level NMHV amplitude is much more complicated than the corresponding tree-level MHV amplitude.

We show one example of NMHV amplitude, $A(1^- 2^- 3^- 4^+ 5^+ 6^+)$. Using BCFW, we again consider the shift

$$p_1(z) = 1(\tilde{1} - z\tilde{6}) \tag{34}$$

$$p_6(z) = (6 + z1)\tilde{6} \tag{35}$$

- The first BCFW bridge is for the separation $\{p_1(z)^-, 2^-, 3^-, 4^+, P(z)^+\}$ and $\{-P(z)^-, 5^+, p_6(z)^+\}$. The right kinematics table is,

$$\begin{array}{c|c|c}
5 & 5 & \tilde{5} \\
\hline
p_6(z) & x_1 5 & \tilde{6} \\
\hline
-P(z) & (-1)5 & 5 + x_1 6
\end{array} \tag{36}$$

It is easy to see that $x_1 = \langle 16 \rangle / \langle 15 \rangle$ and for the case the solution $z_1 = \langle 56 \rangle / \langle 15 \rangle$. The left kinematics table is,

$$\begin{array}{c|c|c}
p_1(z) & 1 & \tilde{1} - z_1\tilde{6} \\
\hline
2 & 2 & \tilde{2} \\
\hline
3 & 3 & \tilde{3} \\
\hline
4 & 4 & \tilde{4} \\
\hline
P(z) & 5 & 5 + x_1 6
\end{array} \tag{37}$$

So the first BCFW term is

$$\begin{aligned}
A_1 &= \frac{-i}{s_{56}} A_L(p_1(z)^-, 2^-, 3^-, 4^+, P(z)^+) A_R(-P(z)^-, 5^+, p_6(z)^+) \\
&= \frac{-i(\langle 1, 5 \rangle [4, 5] + \langle 1, 6 \rangle [4, 6])^3}{\langle 1, 6 \rangle \langle 5, 6 \rangle [2, 3] [3, 4] (\langle 1, 5 \rangle [1, 2] + \langle 5, 6 \rangle [2, 6]) (s_{15} + s_{16} + s_{56})}
\end{aligned} \tag{38}$$

- The second BCFW bridge is for the separation $\{p_1(z)^-, 2^-, P(z)^+\}$ and $\{3^-, 4^+, 5^+, p_6(z)^+, -P(z)^-\}$. The left kinematics table is,

$p_1(z)$	1	$x_2 \tilde{2}$
2	2	$\tilde{2}$
$P(z)$	$(-1)(x_2 1 + 2)$	$\tilde{2}$

(39)

It is easy to see that $x_2 = [16]/[26]$ and for the case the solution $z_2 = -[12]/[26]$. The right kinematics table is,

3	3	$\tilde{3}$
4	4	$\tilde{4}$
5	5	$\tilde{5}$
6	$6 + z_2 1$	$\tilde{6}$
$P(z)$	$(x_2 1 + 2)$	$\tilde{2}$

(40)

So the first BCFW term is

$$\begin{aligned}
A_2 &= \frac{-i}{s_{56}} A_L(p_1(z)^-, 2^-, P(z)^+) A_R(3^-, 4^+, 5^+, p_6(z)^+, -P(z)^-) \\
&= \frac{-i(\langle 1, 3 \rangle [1, 6] + \langle 2, 3 \rangle [2, 6])^3}{\langle 3, 4 \rangle \langle 4, 5 \rangle [1, 2] [1, 6] (\langle 1, 5 \rangle [1, 2] + \langle 5, 6 \rangle [2, 6]) (s_{12} + s_{16} + s_{26})}
\end{aligned} \tag{41}$$

The final result is that

$$A(1^- 2^- 3^- 4^+ 5^+ 6^+) = A_1 + A_2. \tag{42}$$

It is difficult to simplify this amplitude further with spinor helicity formalism.

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- [1] R. Britto, F. Cachazo, and B. Feng, Nucl.Phys. **B715**, 499 (2005), hep-th/0412308.
[2] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys.Rev.Lett. **94**, 181602 (2005), hep-th/0501052.